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# On a Volterra equation of the second kind with 'incompressible' kernel

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**Abstract**
Solving the boundary value problems of the heat equation in noncylindrical domains degenerating at the initial moment leads to the necessity of research of the singular Volterra integral equations of the second kind, when the norm of the integral operator is equal to 1. The paper deals with the singular Volterra integral equation of the second kind, to which by virtue of 'the incompressibility' of the kernel the classical method of successive approximations is not applicable. It is shown that the corresponding homogeneous equation when $|\lambda| > 1$ has a continuous spectrum, and the multiplicity of the characteristic numbers increases depending on the growth of the modulus of the spectral parameter $|\lambda|$. By the Carleman-Vekua regularization method (Vekua in Generalized Analytic Functions, 1988) the initial equation is reduced to the Abel equation. The eigenfunctions of the equation are found explicitly. Similar integral equations also arise in the study of spectral-loaded heat equations (Amangaliyeva *et al.* in Differ. Equ. 47(2):231-243, 2011).
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**Keywords:** Volterra integral equation; Abel equation; spectrum; nontrivial solution; characteristic equation; Carleman-Vekua regularization method

## 1 Introduction

Investigation of boundary value problems for the heat equation in noncylindrical domains has wide practical application [1–3]. For example, in the study of thermal regimes of the various electrical contacts there is the necessity to study the processes of heat and mass transfer taking place between the electrodes. After achieving the melting temperature at the contact surface of electrodes there is a liquid metal bridge between these electrodes. When the contacts open this bridge is divided into two parts, *i.e.* the contact material is transferred from one electrode to another, and this leads to the bridging erosion. Ultimately, the smooth surface of contacts is destroyed, which means that their proper operation is violated. The mathematical description of the thermal processes which go with the bridging erosion, leads to solving the boundary value problems for the heat equation in domains with moving boundary, namely in the domains which degenerate into a point at the initial moment. Using the apparatus of heat potentials, solving the problems under consideration is reduced to the study of singular Volterra integral equations of the second kind, when the norm of the integral operator is equal to 1. A feature of these equations is the incompressibility of the kernel and this is expressed in the fact that the corresponding nonhomogeneous equation cannot be solved by classical methods.

For the problem of the solvability of the Volterra integral equation of the second kind with a special kernel, stated in Section 2, after some transformation in Section 3 we obtain the corresponding characteristic integral equation. An important moment of our research is fact that using a Carleman-Vekua regularization method [4], we reduce the initial problem to solving the Abel integral equation of the second kind. The solution of the last equation provides finding all solutions of the initial integral equation from Section 2. These results are stated in Sections 4-6. The main result about solvability of the integral equation in a class of essentially bounded functions is formulated in the form of the theorem in Section 6.

## 2 Statement of the problem

When solving model problems for parabolic equations in domains with moving boundary the singular integral equations of the following form arise:

$$\varphi(t) - \lambda \int_0^t K(t,\tau)\varphi(\tau)\,d\tau = f(t), \quad t > 0, \tag{1}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{t+\tau}{(t-\tau)^{3/2}} \exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \frac{1}{(t-\tau)^{1/2}} \exp\left(-\frac{t-\tau}{4a^2}\right) \right\}.$$

The kernel $K(t,\tau)$ has the following properties:

(1) $K(t,\tau) \geq 0$ and continuously at $0 < \tau < t < +\infty$;

(2) $\lim_{t\to t_0} \int_{t_0}^t K(t,\tau)\,d\tau = 0$, $t_0 \geq \varepsilon > 0$;

(3) $\lim_{t\to 0} \int_0^t K(t,\tau)\,d\tau = 1$, $\lim_{t\to +\infty} \int_0^t K(t,\tau)\,d\tau = 1$.

To verify property (3) we make the substitution $x = \sqrt{t-\tau}$. We have

$$\begin{aligned}
\int_0^t K(t,\tau)\,d\tau &= -\frac{2}{\sqrt{\pi}} \exp\left\{\frac{2t}{a^2}\right\} \int_0^{\sqrt{t}} \exp\left\{-\left(\frac{t}{ax} + \frac{x}{2a}\right)^2\right\} d\left(\frac{t}{ax} + \frac{x}{2a}\right) \\
&\quad + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \exp\left\{-\frac{x^2}{4a^2}\right\} d\left(\frac{x}{2a}\right) \\
&= \exp\left\{\frac{2t}{a^2}\right\} \operatorname{erfc}\left(\frac{3\sqrt{t}}{2a}\right) + \operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right).
\end{aligned} \tag{2}$$

From (2) the validity of property (3) directly follows. Moreover, it also follows that the norm of the integral operator in (1), acting in the class of essentially bounded functions, is equal to 1.

Also the kernel $K(t,\tau)$ is summable with weight function $t^{-1/2}$. Indeed,

$$\begin{aligned}
\int_0^t \frac{K(t,\tau)}{\sqrt{\tau}}\,d\tau &= \frac{\exp\{2t/a^2\}}{a\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{2\sqrt{t-x^2}}{x^2} \exp\left\{-\left(\frac{t}{ax} + \frac{x}{2a}\right)^2\right\} dx \\
&\quad + \frac{1}{a\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{1}{\sqrt{t-x^2}} \exp\left\{-\left(\frac{t}{ax} - \frac{x}{2a}\right)^2\right\} dx \\
&\quad + \frac{1}{a\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{1}{\sqrt{t-x^2}} \exp\left\{-\frac{x^2}{4a^2}\right\} dx \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

For the first integral after introducing the replacement $y = 1/x$ we have the estimate

$$I_1(t) \leq \frac{2\sqrt{t}\exp\{t/a^2\}}{a\sqrt{\pi}} \int_{t^{-1/2}}^{+\infty} \exp\left\{-\frac{t^2y^2}{a^2} - \frac{1}{4a^2y^2}\right\} dy.$$

For small values $t$ the last integral is bounded. For large values $t \gg 0$ we have the following estimate:

$$\frac{2\sqrt{t}\exp\{t/a^2\}}{a\sqrt{\pi}} \int_{0}^{+\infty} \exp\left\{-\frac{t^2y^2}{a^2} - \frac{1}{4a^2y^2}\right\} dy = \frac{1}{\sqrt{t}} \leq \text{const.}$$

Thus we have established the boundedness of the first integral $I_1(t)$.

We estimate the integrals $I_2(t)$ and $I_3(t)$ using the following integral:

$$\frac{1}{a\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{dx}{\sqrt{t-x^2}} = \frac{1}{2a\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{dy}{\sqrt{y(t-y)}} = \frac{\sqrt{\pi}}{2a}.$$

**Problem** To find the solution $\varphi(t)$ of integral equation (1) satisfying the condition $\sqrt{t} \cdot \varphi(t) \in L_\infty(0,\infty)$ for any given function $\sqrt{t} \cdot f(t) \in L_\infty(0,\infty)$ and each given complex spectral parameter $\lambda \in \mathcal{C}$.

We note that the integral equations of the form (1) arise in the study of boundary value problems of heat conduction in an infinite angular domain, which degenerates at the initial moment. Such equations are called by us Volterra integral equations with 'incompressible' kernel. The feature of the equation in question consists in property (3) of the kernel $K(t,\tau)$ and is expressed in the fact that the corresponding nonhomogeneous equation cannot be solved by the method of successive approximations for $|\lambda| > 1$. Obviously, if $|\lambda| < 1$, then (1) has a unique solution, which can be found by the method of successive approximations. The case $\lambda = 1$ was considered in [5], where it is shown that (1) has only one nontrivial solution at $f(t) \equiv 0$ (within a constant factor). Further in this paper, we assume that $|\lambda| > 1$.

The equations of the form (1) were first considered by SN Kharin: the asymptotics of integrals of the double layer potentials were studied, and approximate solutions of some applied problems were constructed [6, 7]. Subsequently such integral equations were the subject of investigation by many authors.

It should be noted that the boundary value problems for spectrally loaded parabolic equations also are reduced to the singular integral equations under consideration when the load line moves by the law $x = \alpha(t)$ [8, 9].

## 3 Transforming the integral equation

We will use a Carleman-Vekua regularization method [4]. To do this we transform (1). By means of the relations

$$t + \tau = 2t - (t - \tau), \qquad \frac{(t+\tau)^2}{4a^2(t-\tau)} = \frac{t\tau}{a^2(t-\tau)} + \frac{t-\tau}{4a^2},$$

we reduce (1) to the form

$$\varphi(t)-\int_0^t \frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}-\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right.$$
$$\left.+\frac{1}{(t-\tau)^{1/2}}\right\}\cdot\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau=f(t). \tag{3}$$

From [10], p.183, it follows that it suffices to find a solution to the 'simplified' equation

$$\tilde{\varphi}(t)-\lambda\int_0^t k(t,\tau)\tilde{\varphi}(\tau)\,d\tau=\tilde{f}(t), \tag{4}$$
$$\tilde{\varphi}(t)=\exp\{t/(4a^2)\}\cdot\varphi(t),\qquad \tilde{f}(t)=\exp\{t/(4a^2)\}\cdot f(t),$$

where

$$k(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}+\frac{1}{(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\right\},$$
$$\begin{cases}\sqrt{t}\cdot\exp\{-t/(4a^2)\}\cdot\tilde{\varphi}(t)\in L_\infty(0,\infty),\\ \sqrt{t}\cdot\exp\{-t/(4a^2)\}\cdot\tilde{f}(t)\in L_\infty(0,\infty),\\ \sqrt{t}\cdot\exp\{-(t-\tau)/(4a^2)\}\cdot k(t,\tau)\in L_1(0,\infty).\end{cases} \tag{5}$$

To investigate the full equation (4) we will extract its characteristic part, namely

$$\tilde{\varphi}(t)-\lambda\int_0^t k_0(t,\tau)\tilde{\varphi}(\tau)\,d\tau=f_1(t), \tag{6}$$

where

$$k_0(t,\tau)=\frac{t}{a\sqrt{\pi}(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\},$$
$$k_h(t,\tau)=\frac{1}{2a\sqrt{\pi}(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right),$$
$$f_1(t)=\tilde{f}(t)+\lambda\int_0^t k_h(t,\tau)\tilde{\varphi}(\tau)\,d\tau. \tag{7}$$

Equation (6) is characteristic for (4), since

$$\lim_{t\to 0}\int_0^t k_0(t,\tau)\,d\tau=1;\qquad \lim_{t\to 0}\int_0^t k_h(t,\tau)\,d\tau=0.$$

## 4 Solving the characteristic integral equation

Considering the right side of (6) as known, we find its solution, *i.e.* the solution of the characteristic equation (6).

Analogously ([10], p.174) integral equation (6) is reduced to an equation with a difference kernel. To do this, we make in it replacements:

$$t=\frac{1}{y};\qquad \tau=\frac{1}{x};\qquad \psi(y)=\frac{1}{\sqrt{y}}\cdot\tilde{\varphi}\left(\frac{1}{y}\right);\qquad f_2(y)=\frac{1}{\sqrt{y}}\cdot f_1\left(\frac{1}{y}\right). \tag{8}$$

Then we obtain the equation of the form

$$\psi(y) - \lambda \int_y^\infty \frac{1}{a\sqrt{\pi}(x-y)^{3/2}} \exp\left\{-\frac{1}{a^2(x-y)}\right\} \psi(x)\,dx = f_2(y), \quad y > 0, \tag{9}$$

where

$$\begin{cases} \exp\{-1/(4a^2 y)\} \cdot \psi(y) \in L_\infty(0,\infty), \\ \exp\{-1/(4a^2 y)\} \cdot f_2(y) \in L_\infty(0,\infty). \end{cases} \tag{10}$$

The solution of (9) can be found either by the operational method, or by its reduction to Riemann boundary value problem [10, 11].

If we denote $L[\psi(y)] = \overline{\psi}(p)$ as the Laplace transformation of the function $\psi(y)$, then the following formula holds for the convolution:

$$L\left[\int_y^\infty K(y-x)\psi(x)\,dx\right] = \overline{K}(-p)\overline{\psi}(p), \tag{11}$$

where

$$\overline{K}(-p) = \int_0^\infty K(-t)\exp\{pt\}\,dt.$$

As

$$L\left[\frac{b}{2\sqrt{\pi}t^{3/2}} \exp\left\{-\frac{b^2}{4t}\right\}\right] = \exp\{-b\sqrt{p}\}, \quad b = \text{const},$$

then, by virtue of (11), (9) is transformed to

$$\overline{\psi}(p) \cdot \left(1 - \lambda \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right) = \overline{f}_2(p). \tag{12}$$

The corresponding homogeneous equation has the form

$$\overline{\psi}(p) \cdot \left(1 - \lambda \exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right) = 0. \tag{13}$$

In the case when

$$1 - \lambda \exp\left\{-\frac{2}{a}\sqrt{-p}\right\} = 0, \tag{14}$$

the nonzero solutions of (13) are

$$\overline{\psi}_k(p) = C_k \cdot \delta(p - p_k),$$

where $\delta(x)$ is the delta-function, $C_k = \text{const}$, and $p_k$ ($k = 0, \pm1, \pm2, \dots$) are roots of (14). Applying to the last equality the inverse Laplace transformation, we obtain

$$\psi(y) = \frac{1}{2\pi i} \int_{\sigma+i\infty}^{\sigma+i\infty} \delta(p - p_k)\exp\{py\}\,dp = \exp\{p_k y\}$$

(the integral is taken along any straight line $\mathrm{Re}\,p = \sigma$ and understood in the sense of the principal value).

Therefore, if $p = p_k$ are roots of (14), then the eigenfunctions of (9) will have the form [11]

$$\psi_k(y) = C_k \exp\{p_k y\}, \quad C_k = \text{const.} \tag{15}$$

We shall find the roots of (14). When $|\lambda| \geq 1$ we have $\exp\{\frac{2}{a}\sqrt{-p}\} = \lambda$ [11]. Taking the logarithm, we obtain

$$\begin{aligned} &\frac{2}{a}\sqrt{-p} = \ln|\lambda| + i(\arg\lambda + 2k\pi); \quad k = 0, 1, 2, \ldots, \\ &-p_k = \frac{a^2}{4}\bigl(\ln^2|\lambda| - (\arg\lambda + 2k\pi)^2\bigr) + i\frac{a^2}{4}\ln|\lambda|^2(\arg\lambda + 2k\pi). \end{aligned} \tag{16}$$

For the boundedness of functions (15) at infinity it is necessary that $\mathrm{Re}(-p_k) \geq 0$, *i.e.* $\ln^2|\lambda| \geq (\arg\lambda + 2k\pi)^2$ or

$$-\ln|\lambda| \leq \arg\lambda + 2k\pi \leq \ln|\lambda|.$$

Hence $-N_1 \leq k \leq N_2$, where

$$N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \qquad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],$$

$N_1 + N_2 + 1$ is number of eigenfunctions of (15) and $[a]$ is the integer part of $a$. Obviously, the larger $|\lambda|$, the greater the multiplicity of the eigenfunctions.

Thereby, $\forall\lambda$, $|\lambda| \geq 1$ we have

$$\psi_{\mathrm{hom}}(y) = \sum_{k=-N_1}^{N_2} C_k \exp\{p_k y\}.$$

Using the replacements which are inverse to (8), we obtain the solution of the homogeneous equation (6)

$$\tilde{\varphi}_{\mathrm{hom}}(t) = \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\left\{\frac{p_k}{t}\right\},$$

where $\mathrm{Re}\,p_k \leq 0$ by virtue of (16).

We note that if $\lambda = 1$, then $p_0 = 0$. This case is considered in detail in [5, 11, 12].

We rewrite the nonhomogeneous operator equation in the form

$$\overline{\psi}(p) = \overline{f}_2(p) + \frac{\lambda\exp\{-\frac{2}{a}\sqrt{-p}\}}{1 - \lambda\exp\{-\frac{2}{a}\sqrt{-p}\}}\overline{f}_2(p), \quad \text{at } \mathrm{Re}\,p \leq 0.$$

Introducing the notation

$$\overline{r}_{\lambda-}(p) = \frac{\exp\{-\frac{2}{a}\sqrt{-p}\}}{1 - \lambda\exp\{-\frac{2}{a}\sqrt{-p}\}},$$

we will find the original of this image:

$$r_{\lambda-}(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp\{-\frac{2}{a}\sqrt{-p}\}}{1-\lambda\exp\{-\frac{2}{a}\sqrt{-p}\}}\,dp, \quad \text{where } r_{\lambda-}(y) \equiv 0, \text{if } y > 0.$$

In the last integral we have carried out the integration along the contour, avoiding the points $p_k$, determined by (16), on the left. The integral is understood in the sense of the Cauchy principal value. Since we consider $y \le 0$, we close on the right cutting the half-plane (slit is along the positive real semiaxis). The zeros of the denominator of the function

$$\overline{A}(p) = \frac{\exp\{-\frac{2}{a}\sqrt{-p}\}}{1-\lambda\exp\{-\frac{2}{a}\sqrt{-p}\}}$$

are numbers $p_k$, $k = 0, \pm1, \pm2, \ldots$, which we need to circumvent twice in opposite directions. Therefore, according to [11], pp.85-87, we have

$$r_{\lambda-}(y) = \sum_{n=0}^{\infty} \operatorname*{res}_{p=p_n} \overline{A}(p) = \frac{1}{a\sqrt{\pi}(-y)^{3/2}} \sum_{n=1}^{\infty} \frac{n}{\lambda^n} \exp\left\{-\frac{n^2}{a^2(-y)}\right\}.$$

Thus, the solution of the nonhomogeneous equation (9) has the form ([11], pp.86-87):

$$\psi(y) = f_2(y) + \lambda\int_y^{\infty} r_{\lambda-}(y-x) f_2(x)\,dx + \sum_{k=-N_1}^{N_2} C_k \cdot e^{p_k y}, \quad C_k \text{ a const,} \tag{17}$$

where the resolvent $r_{\lambda-}(y)$ is defined above.

Performing the reverse replacements to (8) into (17), we obtain the solution of the nonhomogeneous equation (6)

$$\tilde{\varphi}(t) = f_1(t) + \lambda\int_0^t r(t,\tau) f_1(\tau)\,d\tau + \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot e^{\frac{p_k}{t}}, \tag{18}$$

where

$$r(t,\tau) = \frac{t}{a\sqrt{\pi}(t-\tau)^{3/2}} \sum_{n=1}^{\infty} \frac{n}{\lambda^n} \exp\left\{-n^2 \frac{t\tau}{a^2(t-\tau)}\right\}. \tag{19}$$

## 5 Reducing the integral equation (4) to Abel equation

We shall now get to solving (4), *i.e.* 'the simplified' variant of the initial equation (1).

Using the formula for the solution of the characteristic equation (18), taking into account (7) for the function $f_1(t)$, we obtain

$$\begin{aligned}\tilde{\varphi}(t) &= \tilde{f}(t) + \lambda\int_0^t \frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\tilde{\varphi}(\tau)\,d\tau + \lambda\int_0^t r(t,\tau) \\ &\quad \times \left(\tilde{f}(\tau) + \lambda\int_0^{\tau} \frac{1}{2a\sqrt{\pi(\tau-\tau_1)}}\left(1-\exp\left\{-\frac{\tau\tau_1}{a^2(\tau-\tau_1)}\right\}\right)\tilde{\varphi}(\tau_1)\,d\tau_1\right) d\tau \\ &\quad + \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\left\{\frac{p_k}{t}\right\}.\end{aligned}$$

Changing the order of integration in the right-hand side of this equation and interchanging the roles of $\tau$ and $\tau_1$, we have

$$\begin{aligned}\tilde{\varphi}(t) = \lambda \int_0^t \biggl\{ \frac{1}{2a\sqrt{\pi(t-\tau)}} \biggl( 1 - \exp\biggl\{ -\frac{t\tau}{a^2(t-\tau)} \biggr\} \biggr) \\ + \lambda \int_\tau^t r(t,\tau_1) \frac{1}{2a\sqrt{\pi(\tau_1-\tau)}} \biggl( 1 - \exp\biggl\{ -\frac{\tau_1\tau}{a^2(\tau_1-\tau)} \biggr\} \biggr) d\tau_1 \biggr\} \tilde{\varphi}(\tau)\,d\tau \\ + \tilde{f}(t) + \lambda \int_0^t r(t,\tau)\tilde{f}(\tau)\,d\tau + \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\biggl\{ \frac{p_k}{t} \biggr\}. \qquad (20)\end{aligned}$$

We compute the inner integral in (20)

$$\begin{aligned}J(t,\tau;\lambda) &= \int_\tau^t r(t,\tau_1) \frac{1}{2a\sqrt{\pi(\tau_1-\tau)}} \biggl( 1 - \exp\biggl\{ -\frac{\tau_1\tau}{a^2(\tau_1-\tau)} \biggr\} \biggr) d\tau_1 \\ &= \frac{t}{2a^2\pi} \sum_{n=1}^\infty \int_\tau^t \frac{n}{\lambda^n (t-\tau_1)^{3/2}\sqrt{(\tau_1-\tau)}} \exp\biggl\{ -n^2 \frac{t\tau_1}{a^2(t-\tau_1)} \biggr\} \\ &\quad \times \biggl( 1 - \exp\biggl\{ -\frac{\tau_1\tau}{a^2(\tau_1-\tau)} \biggr\} \biggr) d\tau_1 \\ &= \frac{t}{2a^2\pi} \sum_{n=1}^\infty \frac{n}{\lambda^n} \bigl[ I_n^{(1)}(t,\tau) - I_n^{(2)}(t,\tau) \bigr], \qquad (21)\end{aligned}$$

where

$$\begin{aligned}I_n^{(1)}(t,\tau) &= \int_\tau^t \frac{1}{(t-\tau_1)^{3/2}\sqrt{(\tau_1-\tau)}} \exp\biggl\{ -n^2 \frac{t\tau_1}{a^2(t-\tau_1)} \biggr\} d\tau_1, \\ I_n^{(2)}(t,\tau) &= \int_\tau^t \frac{1}{(t-\tau_1)^{3/2}\sqrt{(\tau_1-\tau)}} \exp\biggl\{ -n^2 \frac{t\tau_1}{a^2(t-\tau_1)} - \frac{\tau_1\tau}{a^2(\tau_1-\tau)} \biggr\} d\tau_1.\end{aligned}$$

Using the substitution $z = \sqrt{(\tau_1-\tau)/(t-\tau_1)}$ we compute the integrals $I_n^{(1)}(t,\tau)$ and $I_n^{(2)}(t,\tau)$. We will have

$$\begin{aligned}I_n^{(1)}(t,\tau) &= \frac{2}{t-\tau} \exp\biggl\{ -\frac{n^2 t\tau}{a^2(t-\tau)} \biggr\} \int_0^\infty \exp\biggl\{ -\frac{n^2t^2z^2}{a^2(t-\tau)} \biggr\} dz \\ &= \frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}} \exp\biggl\{ -\frac{n^2t\tau}{a^2(t-\tau)} \biggr\}, \\ I_n^{(2)}(t,\tau) &= \frac{2}{t-\tau} \exp\biggl\{ -\frac{(n^2+1)t\tau}{a^2(t-\tau)} \biggr\} \int_0^\infty \exp\biggl\{ -\frac{n^2t^2z^2}{a^2(t-\tau)} - \frac{\tau^2}{a^2(t-\tau)z^2} \biggr\} dz \\ &= \frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}} \exp\biggl\{ -\frac{(n+1)^2t\tau}{a^2(t-\tau)} \biggr\}.\end{aligned}$$

When calculating the integral $I_n^{(2)}(t,\tau)$ the following formula was used ([13], p.321, formula 3.325):

$$\int_0^\infty \exp\biggl\{ -\mu x^2 - \frac{\eta}{x^2} \biggr\} = \frac{\sqrt{\pi}}{2\sqrt{\mu}} \exp\{-2\sqrt{\mu\eta}\}, \quad \mu > 0, \eta > 0.$$

Thus, for the difference $I_n^{(1)}(t,\tau) - I_n^{(2)}(t,\tau)$ we obtain

$$I_n^{(1)}(t,\tau) - I_n^{(2)}(t,\tau) = \frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}}\left(\exp\left\{-\frac{n^2 t\tau}{a^2(t-\tau)}\right\} - \exp\left\{-\frac{(n+1)^2 t\tau}{a^2(t-\tau)}\right\}\right).$$

Substituting the last expression into (21), we obtain

$$\begin{aligned} J(t,\tau;\lambda) &= \frac{1}{2a\sqrt{\pi(t-\tau)}}\sum_{n=1}^{\infty}\frac{1}{\lambda^n}\left(\exp\left\{-\frac{n^2 t\tau}{a^2(t-\tau)}\right\} - \exp\left\{-\frac{(n+1)^2 t\tau}{a^2(t-\tau)}\right\}\right) \\ &= \frac{1}{2a\lambda\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}. \end{aligned}$$

Then (20) can be rewritten as

$$\begin{aligned} \tilde{\varphi}(t) = \lambda\int_0^t &\left\{\frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right) + \frac{1}{2a\sqrt{\pi(t-\tau)}}\right. \\ &\left.\times\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right\}\tilde{\varphi}(\tau)\,d\tau + \tilde{f}(t) + \lambda\int_0^t r(t,\tau)\tilde{f}(\tau)\,d\tau \\ &+ \sum_{k=-N_1}^{N_2} C_k\cdot\frac{1}{\sqrt{t}}\cdot\exp\left\{\frac{p_k}{t}\right\}. \end{aligned}$$

Finally, after introducing the notation

$$\tilde{f}_2(t) = \tilde{f}(t) + \lambda\int_0^t r(t,\tau)\tilde{f}(\tau)\,d\tau, \qquad (22)$$

where $r(t,\tau)$ is defined by (19), we obtain

$$\tilde{\varphi}(t) - \frac{\lambda}{2a\sqrt{\pi}}\int_0^t\frac{\tilde{\varphi}(\tau)}{\sqrt{t-\tau}}\,d\tau = \tilde{f}_2(t) + \sum_{k=-N_1}^{N_2} C_k\cdot\frac{1}{\sqrt{t}}\cdot\exp\left\{\frac{p_k}{t}\right\}, \qquad (23)$$

where the solution and the right side of the integral equation (23) belong to classes (5).

Thus, the initial 'simplified' integral equation (4) is reduced to (23), that is, an Abel integral equation of the second kind.

## 6 Solution of Abel integral equation. The main result

According to [10], p.117, a solution of the Abel equation of the second kind,

$$y(x) + \mu\int_a^x\frac{y(t)}{\sqrt{x-t}}\,dt = g(x),$$

has the form

$$y(x) = G(x) + \pi\mu^2\int_a^x\exp\left[\pi\mu^2(x-t)\right]G(t)\,dt, \qquad (24)$$

where

$$G(x) = g(x) - \mu\int_a^x\frac{g(t)}{\sqrt{x-t}}\,dt.$$

We find the solution of the Abel equation (23) for $\tilde{f}_2(t) = 0$, that is, we will find a solution of the corresponding homogeneous equation (4) for each $k$, $-N_1 \le k \le N_2$ (eigenfunctions). Under this condition, (23) for each $k$, $-N_1 \le k \le N_2$, has the form

$$\tilde{\varphi}_k(t) - \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\tilde{\varphi}_k(\tau)}{\sqrt{t-\tau}}\,d\tau = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\}.$$

The solution of this equation can be written as (see (24))

$$\tilde{\varphi}_k(t) = G_k(t) + \frac{\lambda^2}{4a^2} \int_0^t \exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right) G_k(\tau)\,d\tau,$$

where

$$\begin{aligned} G_k(t) &= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\exp\{\frac{p_k}{t}\}}{\sqrt{\tau(t-\tau)}}\,d\tau \\ &= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right). \end{aligned}$$

In calculating the last integral we use the formulas [13], p.367, formula 3.471 (2); p.1025, formula 9.224. Indeed, according to these formulas, we obtain

$$\begin{aligned} &\frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\exp\{\frac{p_k}{t}\}}{\sqrt{\tau(t-\tau)}}\,d\tau = \frac{\lambda}{2a}\left(\frac{-p_k}{t}\right)^{-1/4} \exp\left\{\frac{p_k}{2t}\right\} \cdot W_{-1/4,1/4}\left(\frac{-p_k}{t}\right), \\ &W_{-1/4,1/4}\left(\frac{-p_k}{t}\right) = \sqrt{\pi}\left(\frac{-p_k}{t}\right)^{1/4} \exp\left\{\frac{-p_k}{2t}\right\} \cdot \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right), \end{aligned}$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function.

The function $G_k(t)$ is bounded for $\forall t \in [0;+\infty)$ at $t \to +\infty$ and $G_k(0) = 0$.

Thus, the eigenfunctions of (4) have the form

$$\begin{aligned} \tilde{\varphi}_k(t) &= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) \\ &\quad + \frac{\lambda^2}{4a^2} \int_0^t \exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right) \cdot \left\{\frac{1}{\sqrt{\tau}} \exp\left\{\frac{p_k}{\tau}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right)\right\} d\tau. \end{aligned}$$

We rewrite the previous function in the form

$$\begin{aligned} \tilde{\varphi}_k(t) &= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) \\ &\quad + \frac{\lambda^2}{4a^2} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \left\{ \int_0^t \exp\left(\frac{-\lambda^2}{4a^2}\tau + \frac{p_k}{\tau}\right) \frac{1}{\sqrt{\tau}}\,d\tau \right. \\ &\quad \left. + \frac{\lambda\sqrt{\pi}}{2a} \int_0^t \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right) \exp\left(-\frac{\lambda^2}{4a^2}\tau\right) d\tau \right\} \end{aligned}$$

or

$$\begin{aligned} \tilde{\varphi}_k(t) &= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) \\ &\quad + \frac{\lambda^2}{4a^2} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \left\{ I_{1k}(t;\lambda) + \frac{\lambda\sqrt{\pi}}{2a} I_{2k}(t;\lambda) \right\}, \end{aligned} \tag{25}$$

where

$$I_{1k}(t;\lambda) = \int_0^t \exp\left(-\frac{\lambda^2}{4a^2}\tau + \frac{p_k}{\tau}\right)\frac{1}{\sqrt{\tau}}\,d\tau,$$
$$I_{2k}(t;\lambda) = \int_0^t \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right)\cdot\exp\left(-\frac{\lambda^2}{4a^2}\tau\right)d\tau.$$

After the replacement $z=\sqrt{\tau}$ the integral $I_{1k}(t;\lambda)$ can be written as

$$I_{1k}(t;\lambda) = 2\int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)dz.$$

We compute the integral $I_{2k}(t;\lambda)$ integrating by parts:

$$\left\| \begin{matrix} u = \operatorname{erfc}(\frac{\sqrt{-p_k}}{\sqrt{\tau}}); & dv = \exp(-\frac{\lambda^2}{4a^2}\tau)\,d\tau \\ du = \frac{\sqrt{-p_k}}{\sqrt{\pi}\tau^{3/2}}\exp\{\frac{p_k}{\tau}\}\,d\tau; & v = -\frac{4a^2}{\lambda^2}\exp(-\frac{\lambda^2}{4a^2}\tau) \end{matrix} \right\|. \tag{26}$$

Then, using (26), we have

$$\begin{aligned} I_{2k}(t;\lambda) = {}& -\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot\operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) \\ & + \frac{4a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int_0^t \exp\left(-\frac{\lambda^2}{4a^2}\tau + \frac{p_k}{\tau}\right)\frac{1}{\tau^{3/2}}\,d\tau. \end{aligned}$$

After replacing $z=\sqrt{\tau}$ we obtain

$$\begin{aligned} I_{2k}(t;\lambda) = {}& -\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot\operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) \\ & + \frac{8a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)\frac{1}{z^2}\,dz. \end{aligned}$$

After substituting the expressions obtained for $I_{1k}(t;\lambda)$ and $I_{2k}(t;\lambda)$ into (25) we have

$$\begin{aligned} \tilde{\varphi}_k(t) = {}& \frac{1}{\sqrt{t}}\exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a}\operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) \\ & + \frac{\lambda^2}{4a^2}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\left\{2\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)dz\right. \\ & + \frac{\lambda\sqrt{\pi}}{2a}\left[-\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right)\right. \\ & \left.\left. + \frac{8a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)\frac{1}{z^2}\,dz\right]\right\}. \end{aligned}$$

After some simple transformations we obtain

$$\begin{aligned} \tilde{\varphi}_k(t) = {}& \frac{1}{\sqrt{t}}\exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda^2}{2a^2}\exp\left(\frac{\lambda^2 t}{4a^2}\right) \\ & \times\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)\left(1 + \frac{2a\sqrt{-p_k}}{\lambda z^2}\right)dz \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \\
&\quad \times \int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2} z^2 + \frac{p_k}{z^2}\right)\left(\frac{\lambda}{2a} + \frac{\sqrt{-p_k}}{z^2}\right) dz \\
&= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} - \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \int_0^{\sqrt{t}} \exp\left(\frac{p_k}{z^2} - \frac{\lambda^2}{4a^2} z^2\right) d\left(\frac{\sqrt{-p_k}}{z} - \frac{\lambda}{2a} z\right).
\end{aligned}
$$

After the introduction of the replacement $\xi = \frac{\sqrt{-p_k}}{z} - \frac{\lambda}{2a} z$ we obtain

$$
\begin{aligned}
\tilde{\varphi}_k(t) &= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2} - \frac{\lambda\sqrt{-p_k}}{a}\right) \int_{\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}}^{\{\mathrm{IRP}\}} \exp\left(-\xi^2\right) d\xi \\
&= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2 t}{4a^2} - \frac{\lambda\sqrt{-p_k}}{a}\right) \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right),
\end{aligned}
$$

where $\{\mathrm{IRP}\} = \lim_{t\to+0} \frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}} \in \mathcal{C}$ is infinitely remote point, and the expression denoted as ([13], p.890, formula 8.254$^8$)

$$
\operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_{\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}}^{\{\mathrm{IRP}\}} \exp\left\{-\xi^2\right\} d\xi,
$$

is an integral over an open ended contour from the starting point $\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}$ to the infinitely remote point {IRP}.

Thus, the function

$$
\tilde{\varphi}_k(t) = \exp\left\{\frac{p_k}{t}\right\}\left\{\frac{1}{\sqrt{t}} + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\sqrt{-p_k}}{\sqrt{t}} - \frac{\lambda\sqrt{t}}{2a}\right)^2 \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right)\right\} \tag{27}
$$

is an eigenfunction of the 'simplified' equation (4) for each $k$; $-N_1 \le k \le N_2$, where

$$
N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \qquad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],
$$

and $[a]$ is the integer part of $a$.

Then the function

$$
\tilde{\varphi}(t) = \sum_{k=-N_1}^{N_2} C_k \tilde{\varphi}_k(t) \tag{28}
$$

is a solution of the Abel equation (23) for $\tilde{f}_2(t) = 0$, that is, a solution of the 'simplified' homogeneous equation (4), and the functions $\tilde{\varphi}_k(t)$ and values $p_k$ are determined by (27) and (16), respectively.

We note that after multiplying equality (28) by $\exp(-t/(4a^2))$, we obtain the solution of the homogeneous equation corresponding to the original equation (1):

$$
\begin{aligned}
\varphi(t) = \sum_{k=-N_1}^{N_2} C_k &\left\{\frac{1}{\sqrt{t}} \exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right)\right. \\
&\left. + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2 - 1}{4a^2} t - \frac{\lambda\sqrt{-p_k}}{a}\right) \cdot \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right)\right\}.
\end{aligned} \tag{29}
$$

The function $\sqrt{t}\cdot\varphi(t)$ belongs to the space $L_\infty(0,\infty)$. Indeed, we have for the first terms of the sum (29),

$$\exp\left(\frac{p_k}{t}-\frac{t}{4a^2}\right)\in L_\infty(0,\infty).$$

For the second terms in the sum (29) the following inclusions are also valid:

$$\sqrt{t}\cdot\frac{\lambda\sqrt{\pi}}{2a}\exp\left(\frac{\lambda^2-1}{4a^2}t-\frac{\lambda\sqrt{-p_k}}{a}\right)\cdot\operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right)\in L_\infty(0,\infty).$$

Here it is sufficient to take into account that the numbers $p_k$, $k\in[-N_1,N_2]$, are the roots of (14) for each fixed complex spectral parameter $\lambda\in\mathcal{C}$, and to use the asymptotic form of the function $\operatorname{erfc}(z)$ for large values of $z$ ([13], p.890, formula $8.254^8$; [14], p.758). Obviously, there is a limit relation

$$z=\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\to\{\text{IRP}\}\quad\text{at } t\to\infty \text{ and for each } |\lambda|>1.$$

Thus, the following theorem holds.

**Theorem** *The nonhomogeneous integral equation* (1) *is solvable in the class* $\sqrt{t}\cdot\varphi(t)\in L_\infty(0,\infty)$ *for any right-hand side* $\sqrt{t}\cdot f(t)\in L_\infty(0;\infty)$ *and for each* $|\lambda|>1$. *The corresponding homogeneous equation has* $(N_1+N_2+1)$ *eigenfunctions*

$$\begin{aligned}\varphi_k(t)&=\frac{1}{\sqrt{t}}\exp\left(\frac{p_k}{t}-\frac{t}{4a^2}\right)\\&\quad+\frac{\lambda\sqrt{\pi}}{2a}\exp\left(\frac{\lambda^2-1}{4a^2}t-\frac{\lambda\sqrt{-p_k}}{a}\right)\cdot\operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right),\end{aligned}$$

*and the general solution of integral equation* (1) *can be written as*

$$\varphi(t)=F(t)+\frac{\lambda^2}{4a^2}\int_0^t\exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right)F(t)\,d\tau+\sum_{k=-N_1}^{N_2}C_k\varphi_k(t),$$

*where*

$$F(t)=\tilde{f}_2(t)-\frac{\lambda}{2a\sqrt{\pi}}\int_0^t\frac{\tilde{f}_2(\tau)}{\sqrt{t-\tau}}\,d\tau,$$

*and the function* $\sqrt{t}\cdot\exp\{-t/(4a^2)\}\cdot\tilde{f}_2(t)\in L_\infty(0,\infty)$ *is defined by* (22).

## 7 Conclusion

We studied the problems of resolvability of singular Volterra integral equations of the second kind in the space of essentially bounded functions. It is proved that at $|\lambda|>1$ the homogeneous equation which corresponds to (1) has a continuous spectrum, and the multiplicity of the characteristic numbers increases depending on the growth of the modulus of the spectral parameter $|\lambda|$. The initial equation (1) is reduced to the Abel integral equation (23) by the regularization method of Carleman-Vekua [4], which was developed for solving singular integral equations. The eigenfunctions of (1) are found explicitly and their multiplicity depending on the modulus of the characteristic number $|\lambda|$ is found.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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